A Knife-Point Case for Sion and Wolfe's Game

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May 30, 2019

Abstract

Sion and Wolfe's (1957) game is relatively well-known in the game theory literature for possessing almost of all of the properties sufficient for the existence of Nash equilibrium, yet still lacking one. Here we make a slight but specific alteration to their game that allows for the existence of equilibria, in an effort to explore just why equilibria failed to exist in the original case. We then connect our case to several other examples in the literature to show that these games all lack equilibria because they feature what we call *hiding spots*, which are crucial violations of the payoff security properties that have been established as sufficient for equilibrium existence. Since identifying hiding spots in some games is quite easy, relative fully verifying other conditions, this paper therefore serves as another step toward identifying conditions for the (non-)existence of equilibria.

JEL Classification Codes: C62, C72.

Keywords: Discontinuous games; Nash equilibrium; payoff security; reciprocal upper semicontinuity; finite deviation property; existence.

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1 Introduction

Sion and Wolfe (1957) present a game of conflict between two players, X and Y, on the unit square. The players simultaneously choose $x \in [0,1]$ and $y \in [0,1]$, with payoffs specified as

$$U_x(x,y) = \begin{cases} -1 & \text{if } x < y < x + \frac{1}{2} \\ 0 & \text{if } x = y \text{ or } y = x + \frac{1}{2} \\ +1 & \text{otherwise} \end{cases} \tag{1}$$

for Player X and $U_y(x,y) = -U_x(x,y)\ \forall (x,y) \in [0,1]^2$, so the game is zero-sum. Figure 1 depicts the payoff function $U_x(x,y)$.

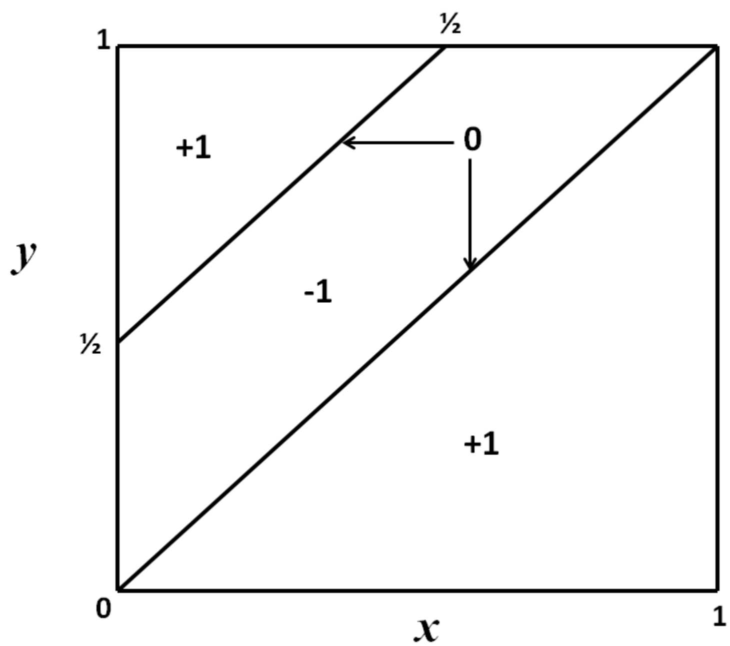

Figure 1: $U_x(x,y)$ from Sion and Wolfe (1957).

In addition to having possible interpretations as a game of strategic conflict, with strategies perhaps representing battle field locations and the payoff function reflecting various tactical advantages of the players, Sion and Wolfe (1957) is also especially relevant for the literature seeking sufficient and/or necessary conditions for equilibrium existence in discontinuous games. This is because the game was originally developed to purposely *not* possess an equilibrium. It therefore serves as a classic example useful for verifying negative results; regardless of what other properties it has, the game does not have a Nash equilibrium.[1]

It is this property that we too are interested in, but while most related work searches for conditions that guarantee existence, our long-term goal is to approach the problem from a somewhat different angle, identifying easy-to-verify conditions that guarantee a game will not have an equilibrium. This paper is a step in that direction, as we show that a small but specific change to Sion and Wolfe's (1957) game can allow equilibria to exist. This alone is not immediately surprising, but a key aspect of our "knife-point" case of the game is that it provides insight into why an equilibrium did not exist for the original case. We can then connect this characteristic to other games that do not possess equilibria.

Specifically, we show that simply modifying the payoff function at one atom so that $U_x(1,0) = 0$ allows equilibria to exist, because this point is what we refer to as a *hiding spot*. In the original game, this point violates the *uniform payoff security* property established by Monteiro and Page (2007). Essentially, Player Y can not be "secure enough" because Player X can *hide* at that point. Modifying the game at that point allows it to satisfy uniform payoff security, which is a sufficient condition for the existence of Nash equilibrium. The change we make to the payoff function of Sion and Wolfe (1957) and the presence–or lack thereof–of uniform payoff security, can then be connected to many other games that lack equilibria. In particular, for two-player constant-sum games, it is the presence of a "hiding spot" such that one player can violate the uniform payoff security of the other that ensures non-existence.

In what follows we present our version of the game and explain how it relates to the existing literature. We then show how hiding spots are responsible for the lack of equilibrium existence in many other well-known examples.

[1] In fact, the game does not even have an ϵ-equilibrium, as noted by Dasgupta and Maskin (1986) in one of the seminal works on existence in discontinuous games.

2 A Knife-Point Case

Our version of Sion and Wolfe's game is identical to the original except that $U_x(1,0) = 0$. We illustrate the change in figure 2.

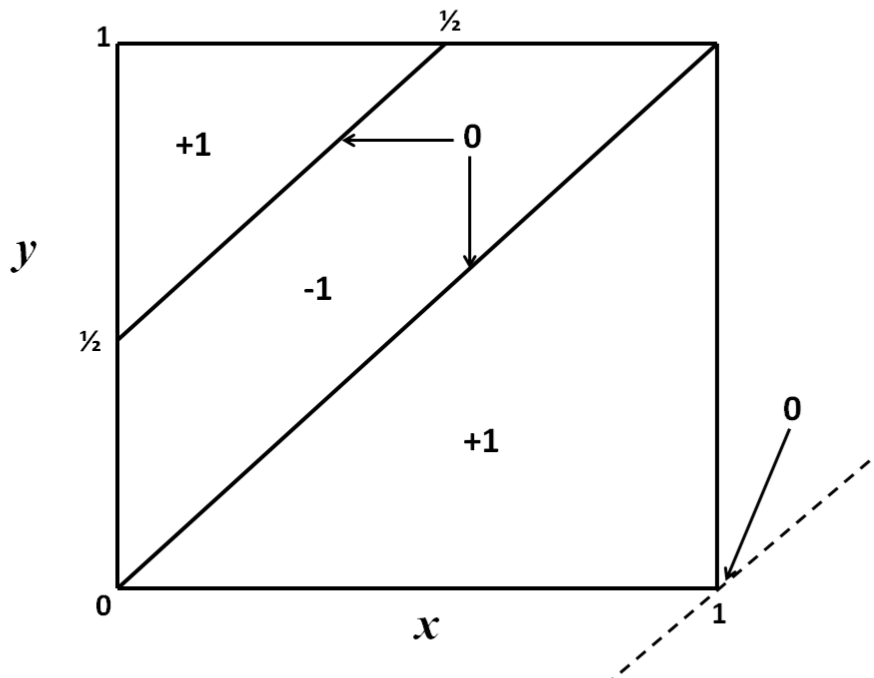

Figure 2: $U_x(x, y)$ for the Knife-Point case.

This version of the game does possess Nash equilibria. In particular, both players employing the mixed strategy f such that

$$f(0) = f(\frac{1}{2}) = f(1) = \frac{1}{3}$$

is an equilibrium. To see this, simply note that

$$U_x(0, f) = U_x(\frac{1}{2}, f) = U_x(1, f) \geq \frac{1}{3},$$

$$U_y(f, 0) = U_y(f, \frac{1}{2}) = U_y(f, 1) \geq -\frac{1}{3},$$

and any unilateral deviation by either player to put weight on points outside the support of f can not improve their payoff.[2] What makes our case interesting is that it further clarifies just why an equilibrium does not exist for the original case.

There have been many advances in terms of establishing sufficient conditions for the existence of equilibria in discontinuous games since Dasgupta and Maskin (1986). Perhaps the most well-known and powerful is the *better-reply security* condition established by Reny (1999), but since this condition can be difficult to verfiy, subsequent work has sought easier to verify or alternative, albeit sometimes more restrictive conditions. Carmona (2011), for example, presents updated results on *weak better-reply security*, and connects his conditions with other reults by Barelli and Meneghal (2013) and updated work by Reny himself (Reny, 2016). The condition most relevant for our example, however, is that of *uniform payoff security*, established by Monteiro and Page (2007). Though it is stricter than Reny's better-reply security, it is easier to verify, and guarantees the existence of a mixed-strategy Nash equilibrium without the complicated process of checking for better-reply security in mixed strategies.

Definition. *Uniform payoff security.* A game with a compact strategy space Z is uniformly payoff secure if each player, starting at any strategy, $z_i \in Z_i$ has a strategy $\hat{z}_i$ they can move to in order to secure a payoff of $u_i(z_i, z_{-i}) - \epsilon$ against deviations by other players in some neighborhood of $_{-i} \in Z_{-i}$ for *all* strategy profiles $_{-i} \in Z_{-i}$ (for each player i and each strategy $_i \in Z_i$ there is a strategy $\hat{z}_i \in Z_i$ that provides security for all z_{-i}, for any $\epsilon > 0$). (Monteiro and Page, 2007)

The crucial result of Monteiro and Page (2007) is that if a game is uniformly payoff secure, its mixed extension is better-reply secure, so it at least has a mixed strategy Nash equilibrium. This is, of course a sufficient condition, and not a necessary one. Nevertheless, note that the original Sion and Wolfe (1957) game is not uniformly payoff secure because of the one point we change. In the origninal version, when $U_x(1, 0) = 1$, Player Y is not uniformly secure. In particular they are not secure at $y = 1$, since they can't secure a payoff of 0 other than at one point. Our version of the game allows them to have one other point to potentially deviate to, removing what we (for the time being) refer to as Player X's *hiding spot*. With

[2]There is actually a continuum of equilibria with the same expected payoffs, since $U(x, f) = 1/3 \; \forall x \in [0, \frac{1}{2})$, so Player X can mix on that interval.

that change, the game is uniformly payoff secure, and equilibria exist.

To see how the same logic holds for similar games, we present a voting game example from Duggan (2007), attributed to Le Breton, which also lacks an equilbirium. It is also a 2-player zero-sum game on the unit square, with the payoff function:

$$U_x(x, y) = \begin{cases} 1 & \text{if } x > y \text{ and either } x < 1 \text{ or} y > 0 \\ 0 & \text{if } x = y \\ -1 & \text{if } x < y \text{ and } x = 0 \text{ and} y = 1 \end{cases} \tag{2}$$

In this case there are two hiding spots that prevent uniform payoff security, the two corners (0,1) and (1,0), and again equilibria will exist if they are changed accordingly.

Heuer (2001) performs a very in-depth study of two-player, three-part partition games on rectangles, with a variety of examples, similar to the Sion and Wolfe game but with the abscissas in a variety of configurations. For each of the cases in which he proves a lack of equilibria, it is quick to see the presence of hiding spots. They are also present in the types of games studied by Prokopovych and Yanellis (2014) in their Theorems 5 and 6–or, more accurately, they would be present if their conditions didn't rule them out, since they establish more general conditions for existence, rather than non-existence. Still, this highlights the importance of hiding spots as an easy identifier for the (non-)existence of equilibria, at least in these types of conflict games.

3 Discussion

We are not the first to attempt to bring equilibria to two-player games on squares, or even to Sion and Wolfe's game in particular. Parthasarathy (1970) presented a version with an equilibrium with the same payoff structure, but restricting at least one player to using absolutely continuous strategies. Duggan (2007) provides sufficient existence conditions for two-player zero-sum games more generally, but again uses conditions that focus on absolutely continuous strategies. Our perspective differs, but is easily reconciled, since the restriction to absolutely continuous strategies, or at least the focus on them to evaluate a game's properties, would eliminate any hiding spots. We feel our perspective is still worthwhile because, although we refer to them as hiding spots, they are easy to spot.

Uniform payoff security is not a necessary condition for the existence of equilibria in general, and the class of games we look at here is narrow. In the future, however, looking for a similar, more general type of hiding spot property may provide another helpful tool for those interested in equilibrium existence.

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